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Examples of Best Discrete l_1 and l_2 Rational Approximations*

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If $r(x)$ is a rational function which approximates a function f given at data points $x_1 < \cdots < x_m$, call it a best l_1 approximation if it provides a local minimum of $\sum_{i=1}^{m} | r(x_i) - f(x_i)|$, and a best l_2 approximation if it provides a local minimum of $\sum_{i=1}^{m} [r(x_i) - f(x_i)]^2$. Let R_{pq} denote the class of rational functions with numerators of degree $\leqslant p$, and denominators ($\neq 0$ on $[x_1, x_m]$) of degree $\leqslant q$. A function $r(x)$ in R_{pq} is degenerate in R_{pq} if it also belongs to $R_{p-k,q-k}$ for some $k > 0$.

A theory for discrete rational l_1 and l_2 approximations is given by Dunham in [1–3], including conditions under which degenerate approximations can be best. However, Dunham failed to give any example of best approximation except for one where 0 is a best L_1 approximation [1, p. 310]. The author gives here an example of nonuniqueness in l_1 (Example 1), an example in l_1 with a nonzero degenerate best approximation (Example 2), and an example in l_2 with the proper number of sign changes (Example 2). These examples should be invaluable in testing algorithms for l_1 and l_2 approximations.

EXAMPLE 1. Given the following data, look for a best l_1 approximation by a function of the form $r(x) = (a_0 + a_1x)/(1 + b_1x)$:

x	0	1	2	3
f	2	1	1	0

Since in this case it is possible to find a function of the class which interpolates the data in three points, one such interpolant, $r(x) = (6 - 2x)/(3 + x)$ is tested to see if it provides a minimum. The constant in the denominator is taken to be 3 to permit all coefficients to be integers.

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To carry out the test, let

$$\tilde{r}(x) = \frac{(6 + \Delta a_0) - (2 + \Delta a_1)x}{3 + (1 + \Delta b_1)x},$$

$\tilde{E}(x) = \tilde{r}(x) - f(x)$ and $E(x) = r(x) - f(x)$, and record $\tilde{E} = E + (\tilde{r} - r)$ at the data points.

x	E
0	$\dfrac{\Delta a_0}{3}$
1	$\dfrac{\Delta a_0 - \Delta a_1 - \Delta b_1}{4 + \Delta b_1}$
2	$\dfrac{\Delta a_0 - 2\Delta a_1 - \frac{4}{5}\Delta b_1}{5 + 2\Delta b_1}$
3	$\dfrac{\Delta a_0 - 3\Delta a_1}{6 + 3\Delta b_1}$

There will be a minimum at $r(x) = (6 - 2x)/(3 + x)$ if for all sufficiently small but otherwise arbitrary choices of Δa_0, Δa_1, Δb_1, the inequality

$$\left|\frac{\Delta a_0}{3}\right| + \left|\frac{\Delta a_0 - \Delta a_1 - \Delta b_1}{4 + \Delta b_1}\right| + \left|\frac{\Delta a_0 - 3\Delta a_1}{6 + 3\Delta b_1}\right| > \left|\frac{\Delta a_0 - 2\Delta a_1 - \frac{4}{5}\Delta b_1}{5 + 2\Delta b_1}\right|$$

is satisfied. Given $\epsilon > 0$, if Δb_1 is chosen to satisfy $|\Delta b_1| < \epsilon$, it will be sufficient to show that

$$\frac{1}{1 + \epsilon}\left[\left|\frac{\Delta a_0}{3}\right| + \left|\frac{\Delta a_0 - \Delta a_1 - \Delta b_1}{4}\right| + \left|\frac{\Delta a_0 - 3\Delta a_1}{6}\right|\right] > \frac{1}{1 - \epsilon}\left|\frac{\Delta a_0 - 2\Delta a_1 - \frac{4}{5}\Delta b_1}{5}\right|.$$

A small computation gives

$$\frac{1}{5}\left(\Delta a_0 - 2\Delta a_1 - \frac{4}{5}\Delta b_1\right) = -\frac{3}{25}\left(\frac{\Delta a_0}{3}\right) + \frac{16}{25}\left(\frac{\Delta a_0 - \Delta a_1 - \Delta b_1}{4}\right) + \frac{12}{25}\left(\frac{\Delta a_0 - 3\Delta a_1}{6}\right).$$

Therefore,

$$\left|\frac{\Delta a_0 - 2\Delta a_1 - \frac{4}{5}\Delta b_1}{5}\right|$$

$$\leqslant \frac{3}{25}\left|\frac{\Delta a_0}{3}\right| + \frac{16}{25}\left|\frac{\Delta a_0 - \Delta a_1 - \Delta b_1}{4}\right| + \frac{12}{25}\left|\frac{\Delta a_0 - 3\Delta a_1}{6}\right|$$

$$< \frac{16}{25}\left[\left|\frac{\Delta a_0}{3}\right| + \left|\frac{\Delta a_0 - \Delta a_1 - \Delta b_1}{4}\right| + \left|\frac{\Delta a_0 - 3\Delta a_1}{6}\right|\right].$$

Since for small ϵ, $16/25 \cdot (1 + \epsilon)/(1 - \epsilon) < 1$, the required inequality is satisfied and $r(x) = (6 - 2x)/(3 + x)$ is a best l_1 approximation. However, $r(x)$ is not unique since $\tilde{r}(x) = (12 - 4x)/(6 - x)$ also provides a minimum.

The final example provides both a degenerate best rational l_1 approximation and a best rational least-squares approximation which does not interpolate the approximated function at any point, although it has the required number of sign changes.

EXAMPLE 2. Given the following data, find best l_1 and l_2 approximations by functions of the form $r(x) = (a_0 + a_1x)/(1 + b_1x)$:

x	0	1	2	3
f	1	2	0	1

Consider first the l_1 approximation. The degenerate member $r(x) = 1$ of the class R_{11} interpolates f at the two end points, and is a candidate for a minimum. Retaining previously used notation, with

$$\tilde{r}(x) = \frac{1 + \Delta a_0 + \Delta a_1 x}{1 + \Delta b_1 x},$$

construct the table

x	f	r	$\tilde{r}$	E	$\tilde{E}$
0	1	1	$\frac{1 + \Delta a_0}{1}$	0	Δa_0
1	2	1	$\frac{1 + \Delta a_0 + \Delta a_1}{1 + \Delta b_1}$	-1	$-1 + \frac{\Delta a_0 + \Delta a_1 - \Delta b_1}{1 + \Delta b_1}$
2	0	1	$\frac{1 + \Delta a_0 + 2\Delta a_1}{1 + 2\Delta b_1}$	1	$1 + \frac{\Delta a_0 + 2\Delta a_1 - 2\Delta b_1}{1 + 2\Delta b_1}$
3	1	1	$\frac{1 + \Delta a_0 + 3\Delta a_1}{1 + 3\Delta b_1}$	0	$\frac{\Delta a_0 + 3\Delta a_1 - 3\Delta b_1}{1 + 3\Delta b_1}$

The function being tested is a minimum if for all sufficiently small Δa_0, Δa_1, Δb_1,

$$\left| \frac{\Delta a_0 + 2\Delta a_1 - 2\Delta b_1}{1 + 2\Delta b_1} - \frac{\Delta a_0 - \Delta a_1 - \Delta b_1}{1 + \Delta b_1} \right| \leqslant |\, \Delta a_0 \,| + \left| \frac{\Delta a_0 + 3\Delta a_1 - 3\Delta b_1}{1 + 3\Delta b_1} \right|.$$

Choose Δb_1 so that $|\, \Delta b_1 \,| < 1/10$. Then

$$\begin{aligned} &\left| \frac{\Delta a_0 + 2\Delta a_1 - 2\Delta b_1}{1 + 2\Delta b_1} - \frac{\Delta a_0 + \Delta a_1 - \Delta b_1}{1 + \Delta b_1} \right| \\ &= \frac{|\, \Delta a_1 - \Delta b_1 - \Delta a_0 \, \Delta b_1 \,|}{|\, 1 + 2\Delta b_1 \,| \cdot |\, 1 + \Delta b_1 \,|} \\ &< 3 \cdot \frac{|\, 1 + 2\Delta b_1 \,| \cdot |\, 1 + \Delta b_1 \,|}{|\, 1 + 3\Delta b_1 \,|} \cdot \frac{|\, \Delta a_1 - \Delta b_1 - \Delta a_0 \, \Delta b_1 \,|}{|\, 1 + 2\Delta b_1 \,| \cdot |\, 1 + \Delta b_1 \,|} \\ &= \frac{3 \,|\, \Delta a_1 - \Delta b_1 - \Delta a_0 \, \Delta b_1 \,|}{|\, 1 + 3\Delta b_1 \,|} = \left| \frac{\Delta a_0 + 3\Delta a_1 - 3\Delta b_1}{1 + 3\Delta b_1} - \Delta a_0 \right| \\ &\leqslant \left| \frac{\Delta a_0 + 3\Delta a_1 - 3\Delta b_1}{1 + 3\Delta b_1} \right| + |\, \Delta a_0 \,|. \end{aligned}$$

The inequality of the test is satisfied and thus $r(x) = 1$ is a local minimum which provides an example of a degenerate best rational l_1 approximation.

The least-squares problem for the same data is that of finding $r(x) = (a_0 + a_1 x)/(1 + b_1 x)$ which minimizes $S = \sum_i (r_i - f_i)^2$. The equations which must be satisfied by a_0, a_1, b_1 in this case are

$$\text{(i)} \quad \sum_i 2 \left[\frac{a_0 + a_1 x_i}{1 + b_1 x_i} - f_i \right] \cdot \left[\frac{1}{1 + b_1 x_i} \right] = 0,$$

$$\text{(ii)} \quad \sum_i 2 \left[\frac{a_0 + a_1 x_i}{1 + b_1 x_i} - f_i \right] \cdot \left[\frac{x_i}{1 + b_1 x_i} \right] = 0,$$

$$\text{(iii)} \quad \sum_i 2 \left[\frac{a_0 + a_1 x_i}{1 + b_1 x_i} - f_i \right] \cdot \left[\frac{-x_i(a_0 + a_1 x_i)}{(1 + b_1 x_i)^2} \right] = 0.$$

These equations are satisfied by $a_0 = 13/10$, $a_1 = -\frac{1}{5}$ and $b_1 = 0$. The function $r(x) = 13/10 - \frac{1}{5}x$ is a member of the class R_{11} having the following table of values:

x	$r(x)$	$f(x)$	$E(x)$	E^2
0	$\frac{13}{10}$	1	$\frac{3}{10}$	$\frac{9}{100}$
1	$\frac{11}{10}$	2	$-\frac{9}{10}$	$\frac{81}{100}$
2	$\frac{9}{10}$	0	$\frac{9}{10}$	$\frac{81}{100}$
3	$\frac{7}{10}$	1	$-\frac{3}{10}$	$\frac{9}{100} \sum E_i^2 = \frac{9}{5}$.

The second derivatives required to test this for a minimum have been calculated as follows: $\delta^2 S/\delta a_0^2 = 8$; $\delta^2 S/\delta a_0\, \delta a_1 = 12$; $\delta^2 S/\delta a_0\, \delta b_1 = -10$; $\delta^2 S/\delta a_1^2 = 28$; $\delta^2 S/\delta a_1\, \delta b_1 = -22$; $\delta^2 S/\delta b_1^2 = 19.16$. The matrix

$$\begin{pmatrix} 8 & 12 & -10 \\ 12 & 28 & -22 \\ -10 & -22 & 19.26 \end{pmatrix}$$

is positive definite and thus the function $r(x)$ that has been found is a minimum. It does not interpolate f at any point; however, it does have three sign changes.

References

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